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ENGINEERING GRAPHICS SEMINAR  
FOUR-DIMENSIONAL DESCRIPTIVE GEOMETRY  
PROBLEMS ON 3-D SPACES

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Princeton, New Jersey.

### ABSTRACT

The paper presents new solutions of five problems in four-dimensional descriptive geometry. These problems were briefly discussed in previous papers. The new solutions have already been used in papers preceding this one but were not fully examined.

1.

### FIRST PROBLEM

"Given a 3-D space  $\Lambda$ , determine the orthographic projections of a line of the 3-D space perpendicular to a plane  $\alpha$ , also of the 3-D space, through a point (a) of the plane."

The first consideration is the question of the determination of a plane  $\alpha$  belonging to the 3-D space  $\Lambda$ . Figure 1 shows a plane (mnp) satisfying the condition, where each point belongs to the traces of  $\Lambda$  in the planes  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$  of the 4-D system of reference.

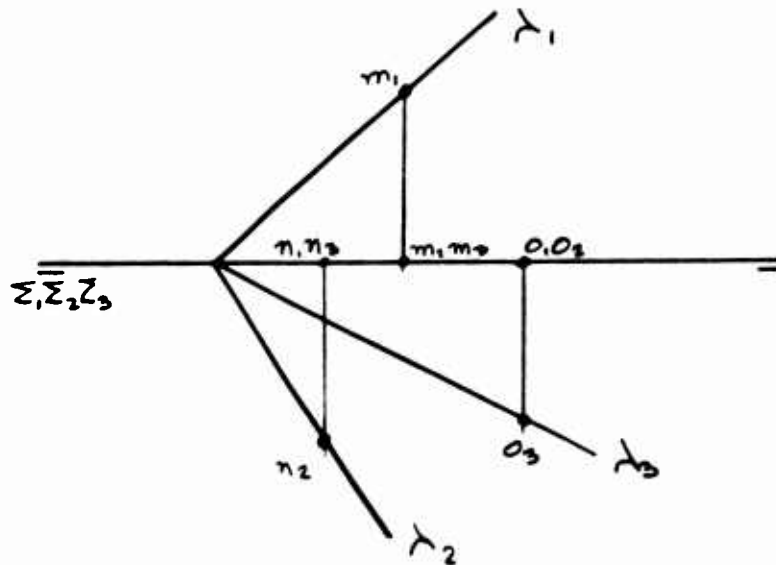


Figure 1

## 2.

The procedure for the determination of the projection of a line perpendicular to this plane is outlined in our paper "Descriptive Geometry of Four Dimensions"<sup>1)</sup> and consists in making changes of 3-D spaces in the 3-D system of reference, so that the space  $\Lambda$  is superimposed on one of those 3-D spaces of the system. Thus, the solution is obtained by methods of the three-dimensional descriptive geometry.

Evidently the above approach may be applied to all other cases, but it is more or less apparent that some "direct" solution may be obtained for some particular cases when the plane occupies a special position in relation to one of the 3-D spaces of the 4-D system of reference. Therefore, the use of a plane in one of these particular positions would not justify the application of the generalized solution.

With this in mind we searched for the particular positions of the plane, within a 3-D space  $\Lambda$ . Those positions may be obtained as results of the intersection of  $\Lambda$  with a 3-D space  $\Omega$  which is a geometric locus. For example, the plane  $\alpha$ , intersection of  $\Lambda$  and  $\Omega$  being 3-D space  $\Omega$  parallel to the 3-D space  $\Sigma_2$  of the system of reference, is a plane parallel to  $\Sigma_2$ . (Figure 2).

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<sup>1)</sup> Engineering Graphics Seminar, Technical Seminar Series, Report No. 9, pp. 29-30, December 5, 1963, Department of Graphics and Engineering Drawing, Princeton University.

**Figure 2**

If we now compare the arrangement of the lines  $(sk, tk)$  of the plane  $\alpha$ , with the arrangement of the lines  $(sk, tk)$  of a plane  $\beta$  (Figure 3) where they belong to a 3-D space, we can contemplate the idea that since it is immediately possible to determine the projections of a line  $(ab)$  perpendicular to  $\beta$ , (Figure 4), it should be also possible to determine the projections of  $(ab)$  perpendicular to  $\alpha$  in the 4-D space, without the use of the generalized solution indicated for the case of the plane in figure 1.

4.

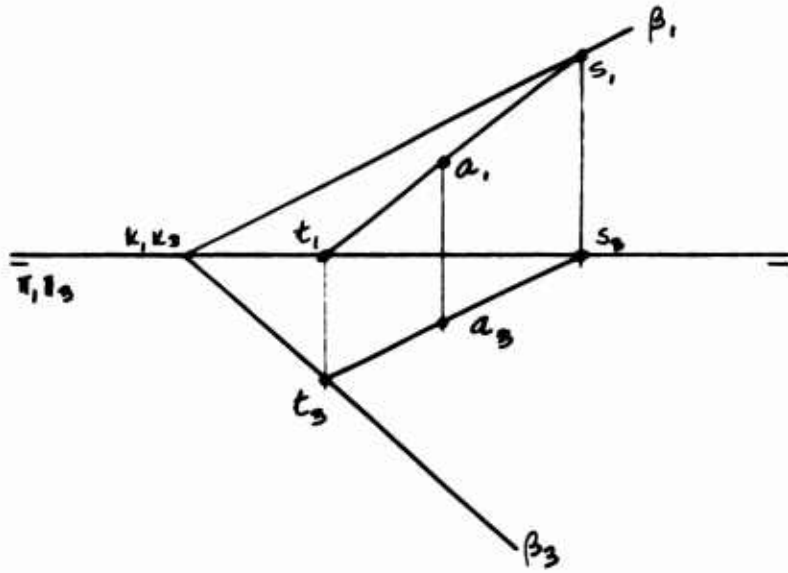


Figure 3

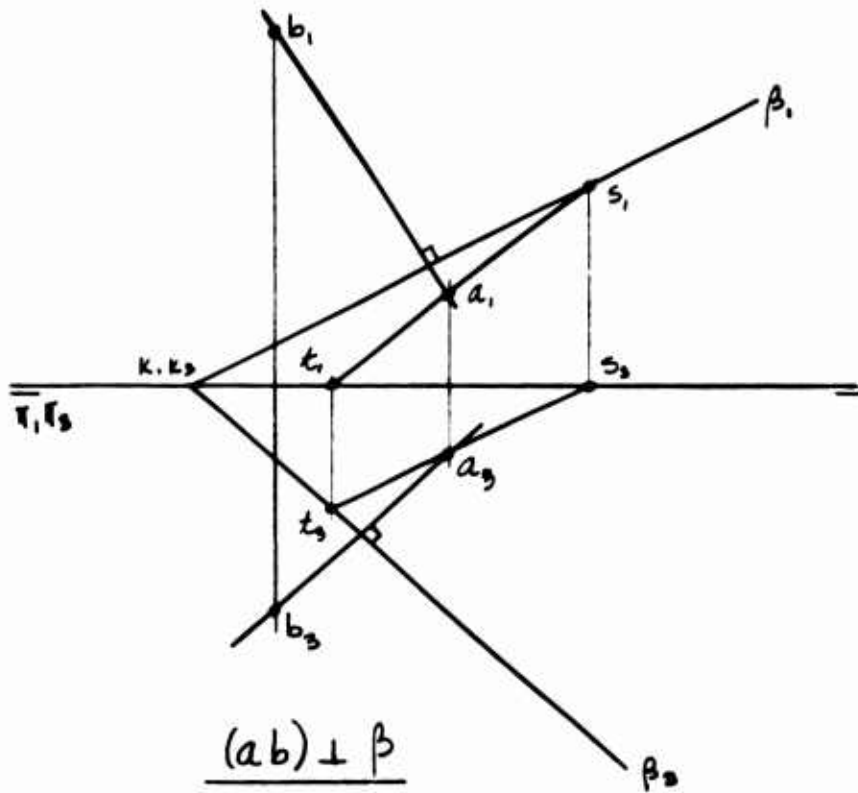


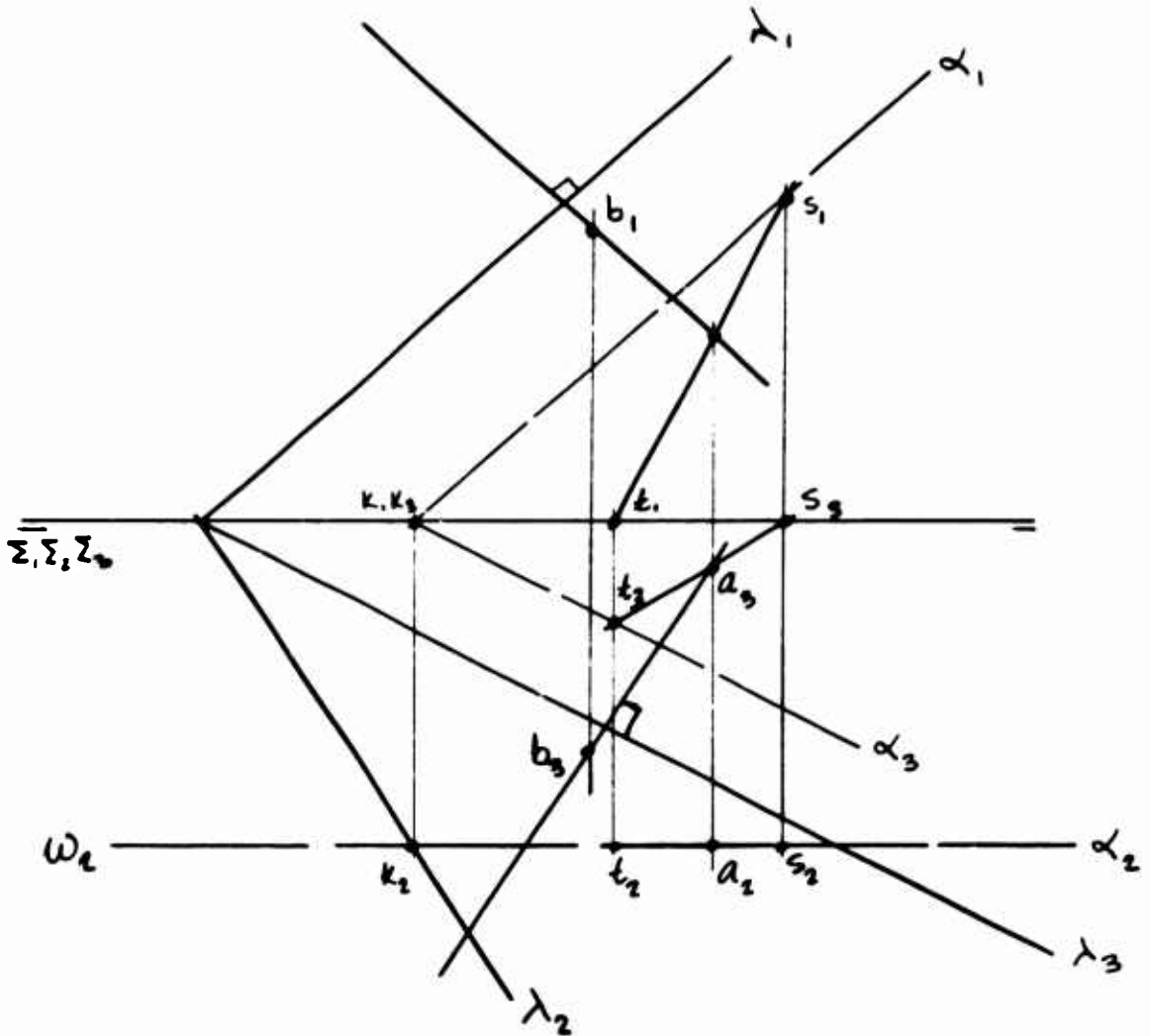
Figure 4

5.

Further analysis leads then to the conclusion that there exists the same relationship, in 4-D descriptive geometry, between the projections of a line perpendicular to a plane and those of the lines (sk) and (st) shown in figure 2, as in the case of the 3-D descriptive geometry shown in figure 4: the projections of the line are perpendicular to the traces of same sub-index, of the plane.

Thus it may be demonstrated that the projections  $a_1b_1$  and  $a_3b_3$  of a line (ab) perpendicular to the plane  $\alpha$  of a 3-D space  $\Lambda$  are perpendicular, respectively, to the projections  $s_1k_1$  and  $s_3t_3$  of two lines (sk) and (st) of that plane, being each one of these lines, parallel to two of the traces of the 3-D space  $\Lambda$ . (See figure 5). The only remaining problem is the determination of the projection  $a_2b_2$  of the line (ab).

6.



$\Lambda \times \Omega \Rightarrow \text{PLANE } \alpha$

$(sk) \text{ AND } (sk) \text{ BELONG TO } \alpha$

$\alpha_1 \equiv (sk) \parallel \lambda_1 ; \alpha_2 \equiv (tk) \parallel \lambda_2$

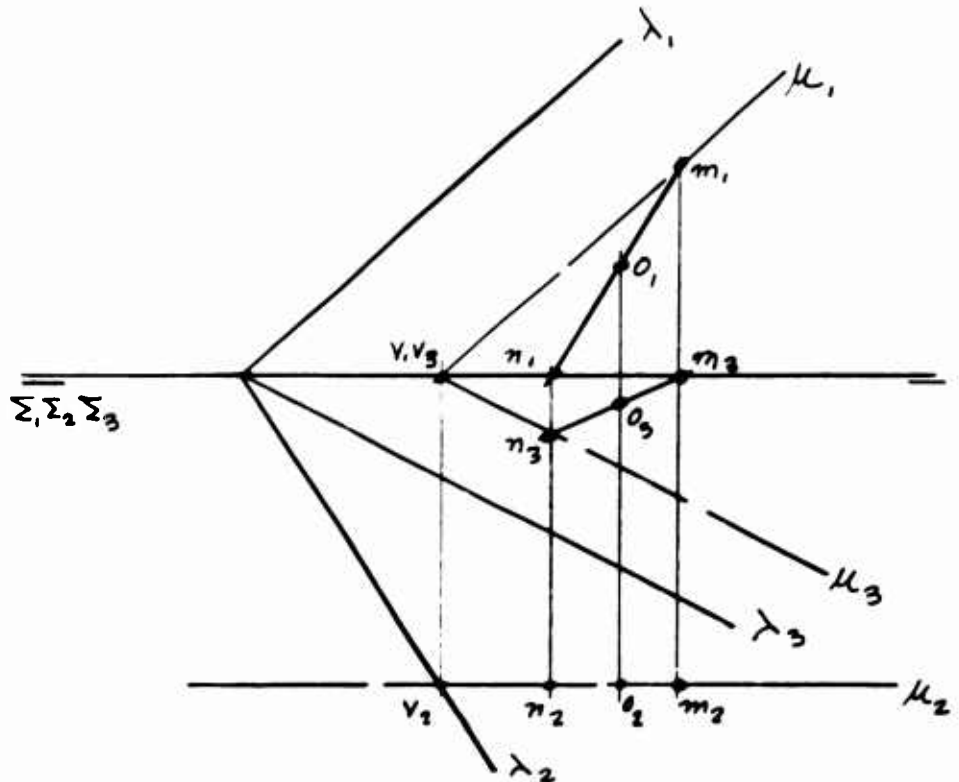
$a_1, b_1 \perp \alpha_1 ; a_2, b_2 \perp \alpha_2$

PROBLEM: DETERMINE  $a_2, b_2$

Figure 5

7.

To determine the projection  $b_2$  of the point (b) we shall make use of the conditions of belonging between a point and a 3-D space: the point belongs to a line of a plane in the 3-D space. Figures 6, 7, 8, show a point (o) of the 3-D space  $\Lambda$ , which condition of belonging is satisfied by use of planes and lines belonging to the 3-D space.



$$\left. \begin{array}{l} (o) \rightarrow (mn) \\ (mn) \rightarrow \Lambda \end{array} \right\} (o) \rightarrow \Lambda$$

Figure 6

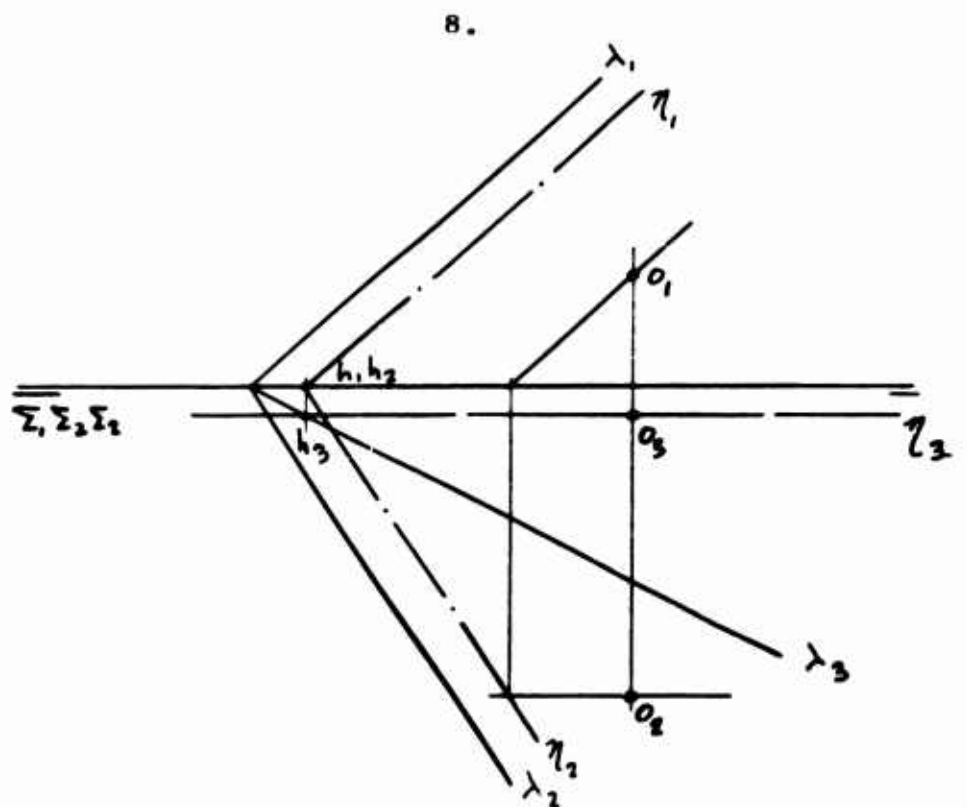


Figure 7

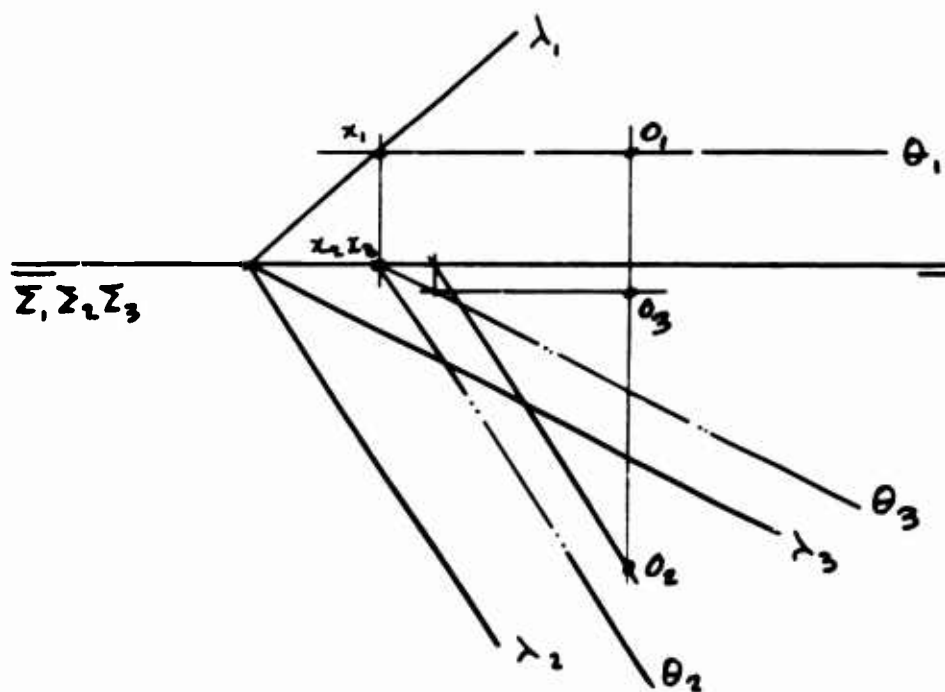
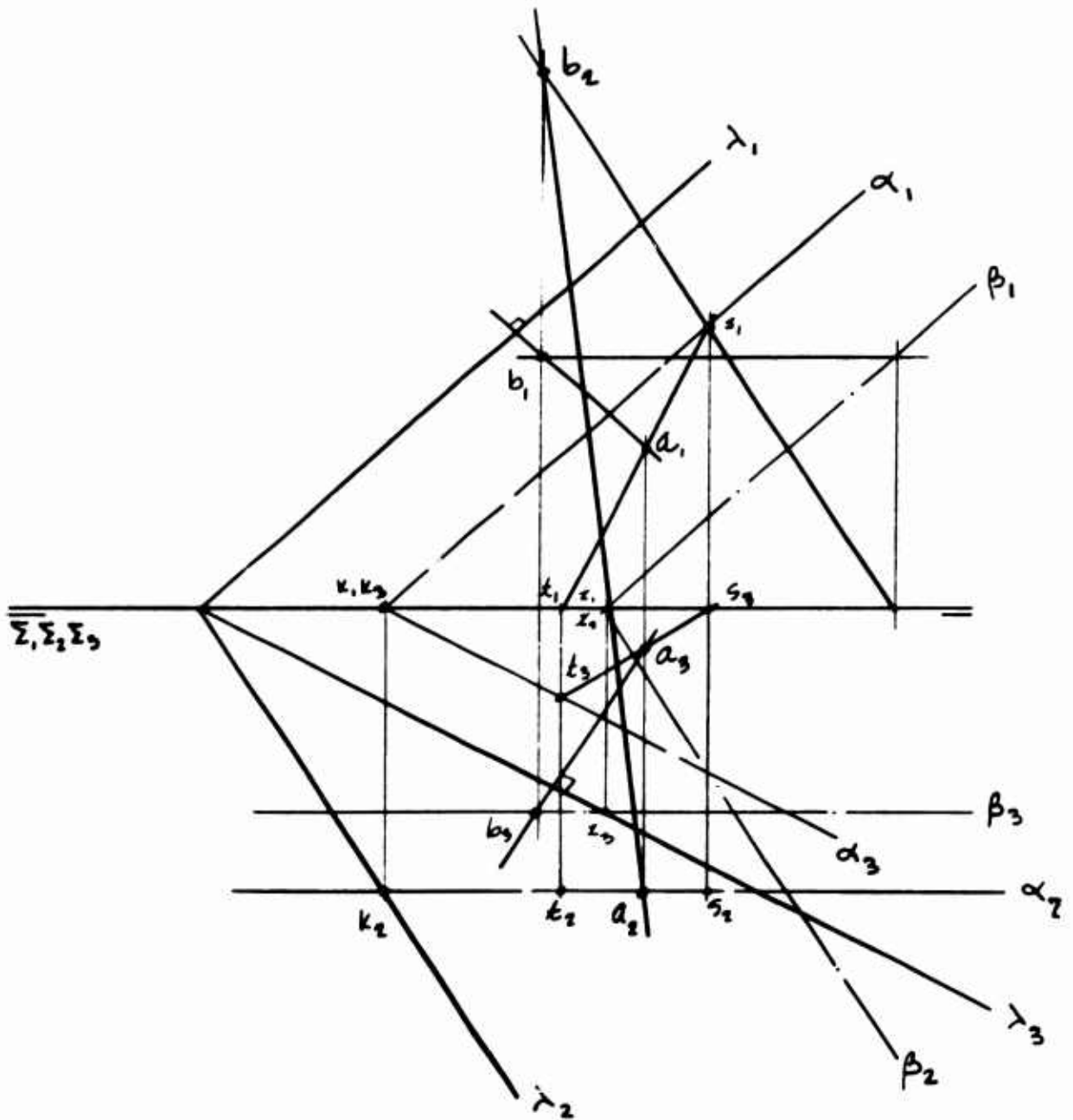


Figure 8

Thus it can be seen that if one or two of the projections of a point are given, it is possible to determine the other projections. This is the case of the point (b), being known  $b_1$  and  $b_3$ . Figure 9 shows how the projection  $b_2$  is determined. Therefore,  $a_1b_1, a_3b_3$  are the projections of a line (ab) of  $\Lambda$ , perpendicular to the plane  $\alpha$ , of  $\Lambda$ , through the point (a) of  $\alpha$ .

**Figure 9**

SECOND PROBLEM

¶Given a 3-D space  $\mathcal{T}$  by its traces and a point (a) determine the traces of the 3-D space  $\Delta$  parallel to  $\mathcal{T}$  and belonging to (a).<sup>2)</sup>

Shown in figure 10 are the given 3-D space  $\mathcal{T}$  and the point (a).

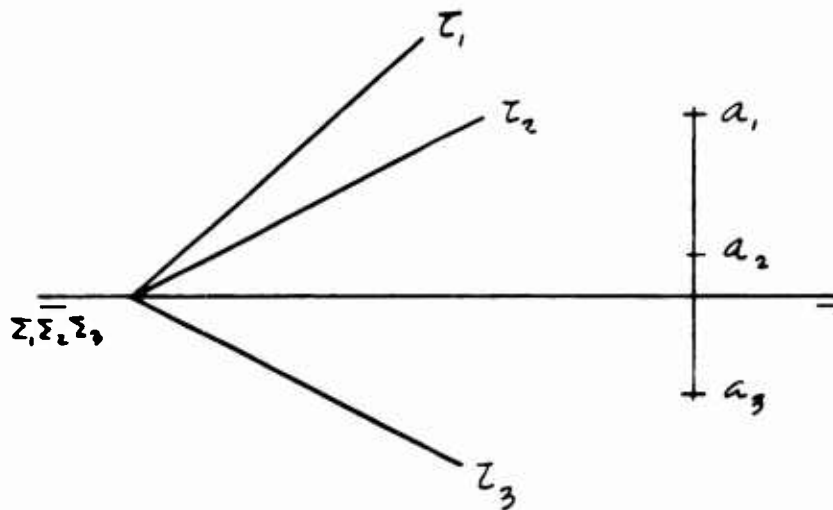


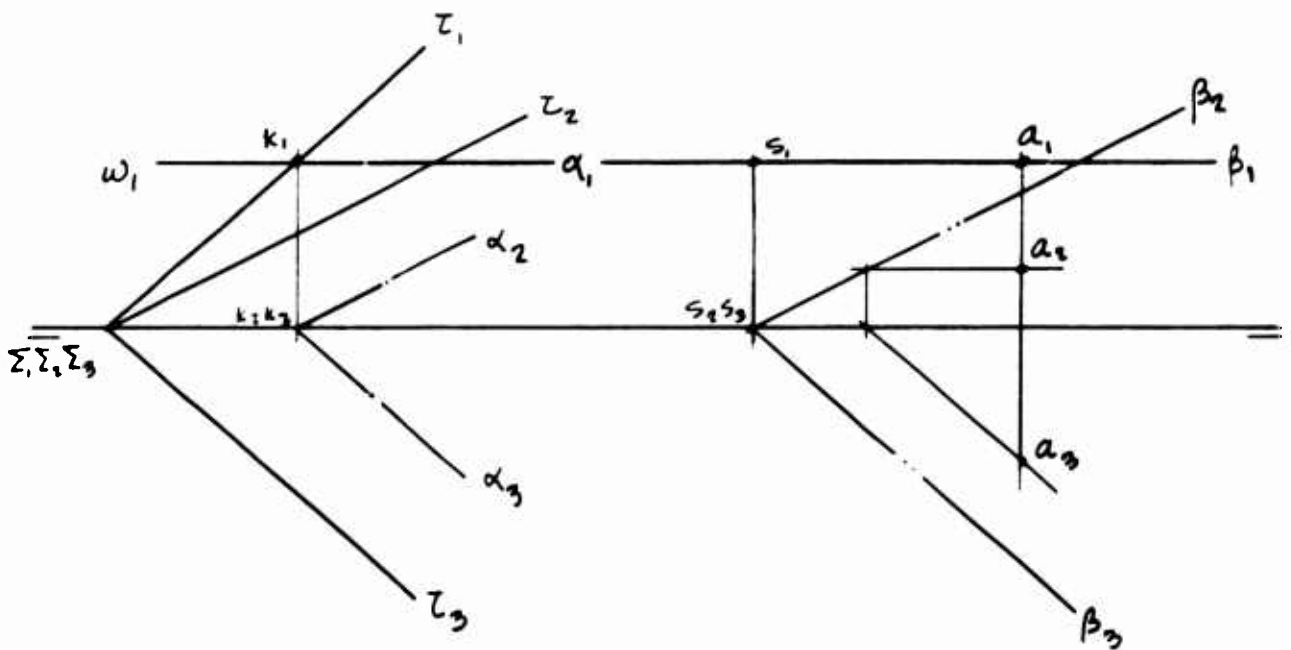
Figure 10

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<sup>2)</sup> Report No. 9, Ibid, pp. 19-20.

### SOLUTION

If through (a) we consider the 3-D space  $\Omega$  parallel to  $\Sigma_1$ , the intersections with the 3-D spaces  $\mathbb{T}$  and  $\Delta$  are the planes  $\alpha$  and  $\beta$ , respectively, shown in figure 11.



**Figure 11**

The 3-D space  $\Lambda$  belongs to  $\beta$  and to the point (s), this point belonging to the trace  $\lambda_1$ . Therefore, to determine the traces of  $\Lambda$ , through  $s_1$  draw parallel to  $\tau_1$ , to obtain  $\lambda_1$  and the point (o) on the reference line. Through

13.

(o), draw parallels to  $\tau_2$  and  $\tau_3$ , to obtain  $\lambda_2$  and  $\lambda_3$ . See figure 12.

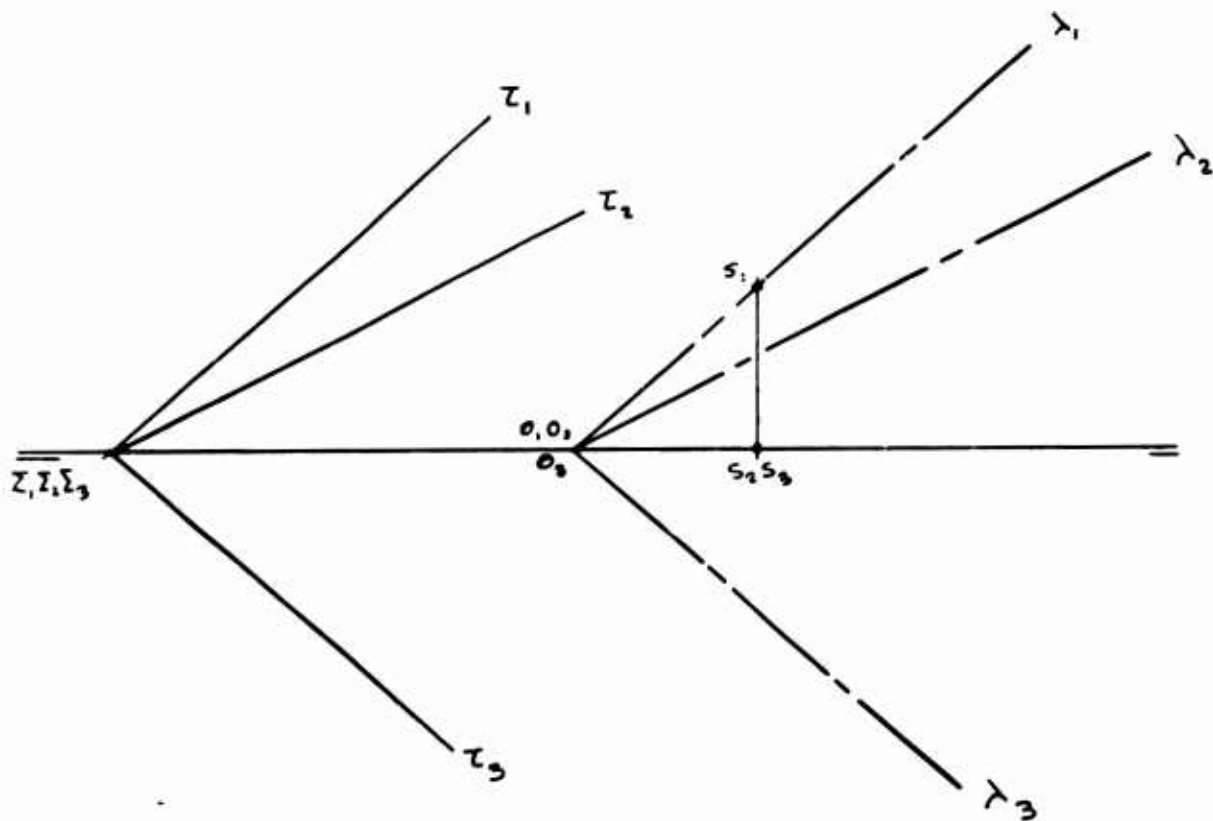


Figure 12

### THIRD PROBLEM

"To rotate a 3-D space  $\Omega$  until its superimposition on one of the 3-D spaces of the 4-D system of reference."

14.

SOLUTION

The rotation is made about a plane of  $\Omega$ , also belonging to the 3-D space of reference.

Let  $\Omega$  be the given 3-D space (figure 13) which is to be rotated until its superimposition on  $\Sigma_2$ .

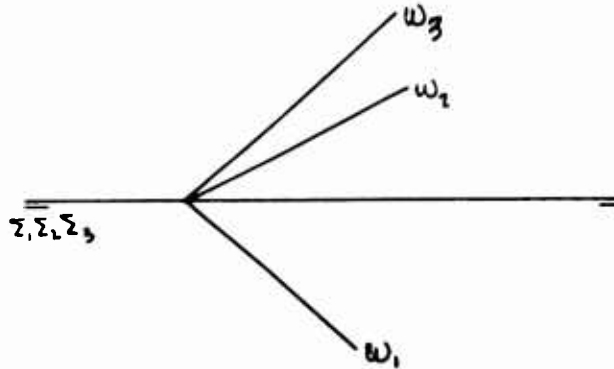


Figure 13

Consider that the three traces,  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  are the edges of a trihedral angle whose faces are the planes  $\omega_1\omega_2$ ,  $\omega_2\omega_3$ ,  $\omega_1\omega_3$ , and whose vertex is (o). Therefore, the point (b) of the edge  $\omega_2$  can be referred to the opposite face  $\omega_1\omega_3$  (in the 3-D space  $\Sigma_2$ ) by its projection on that face and by its distance to it.

The determination of this projection and of that distance are indicated in figures 14 and 15. First we draw

15.

a perpendicular through (b) to the plane  $\omega_1\omega_3$  and determine the foot of the perpendicular, point (p), which is then, the projection of (b) in the plane  $\omega_1\omega_3$ . The true length of (bp) is the distance from (b) to  $\omega_1\omega_3$ .

To determine the projection  $p_3$  of the foot of the perpendicular we made use of a plane  $\rho(\rho_1, \rho_2, \rho_3)$  belonging to (p) and to the 3-D space  $\Omega$ . As expected, the projection  $p_2$  is found on the reference line, for the plane  $\omega_1\omega_3$  belongs to the 3-D space  $\Sigma_2$ .

16.

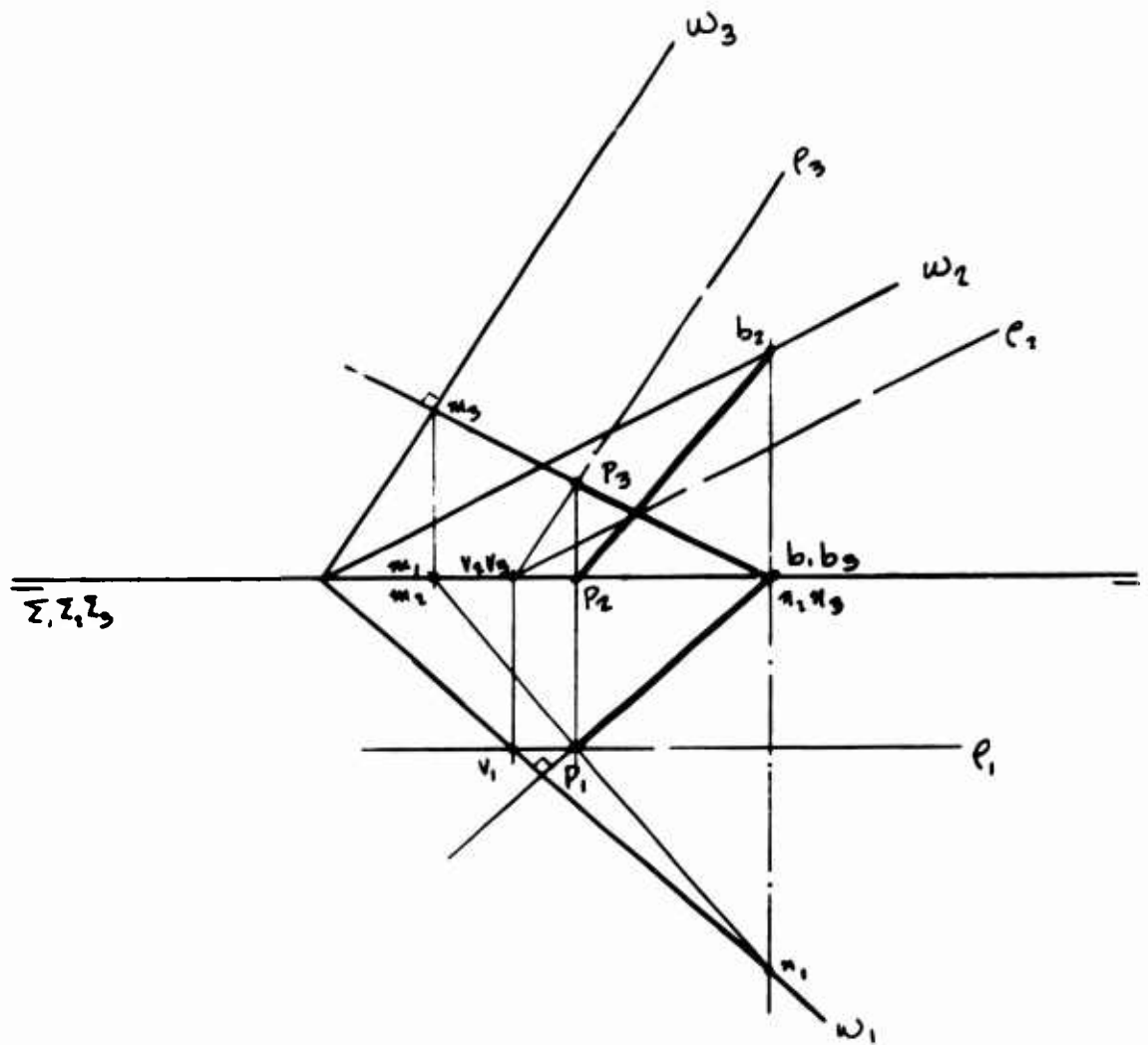
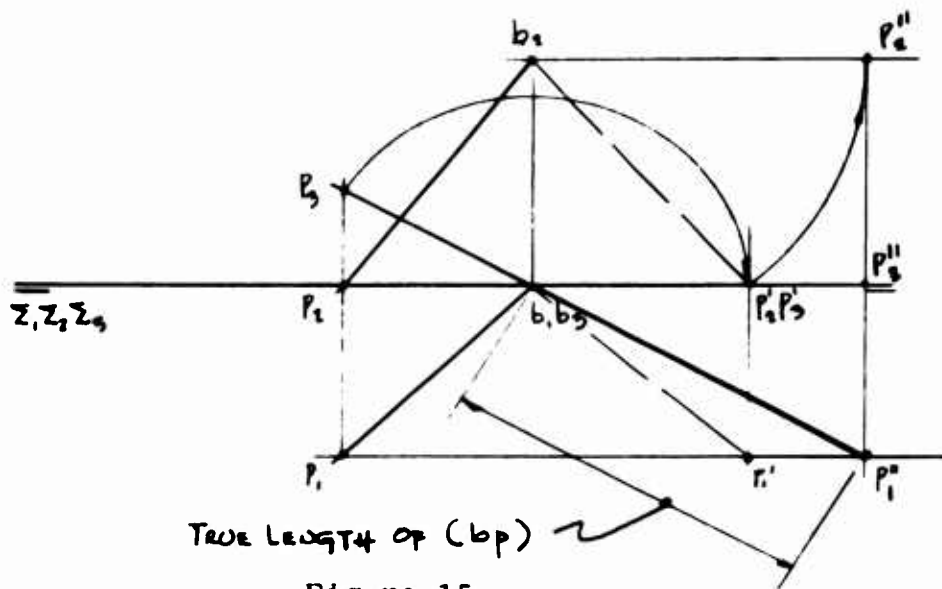


Figure 14

17.



**Figure 15**

To obtain the location of the edge  $\omega_2$  in the 3-D space  $\Sigma_2$ , all we have to do now is to consider the point (p) and the plane  $\omega_1\omega_3$  of that 3-D space, and operate the following construction by three-dimensional descriptive method: from (p) raise perpendicular to  $\omega_1\omega_3$ , in the 3-D space  $\Sigma_2$  and mark a point (s) on this perpendicular so that (ps) = (bp). Figure 16.

18.

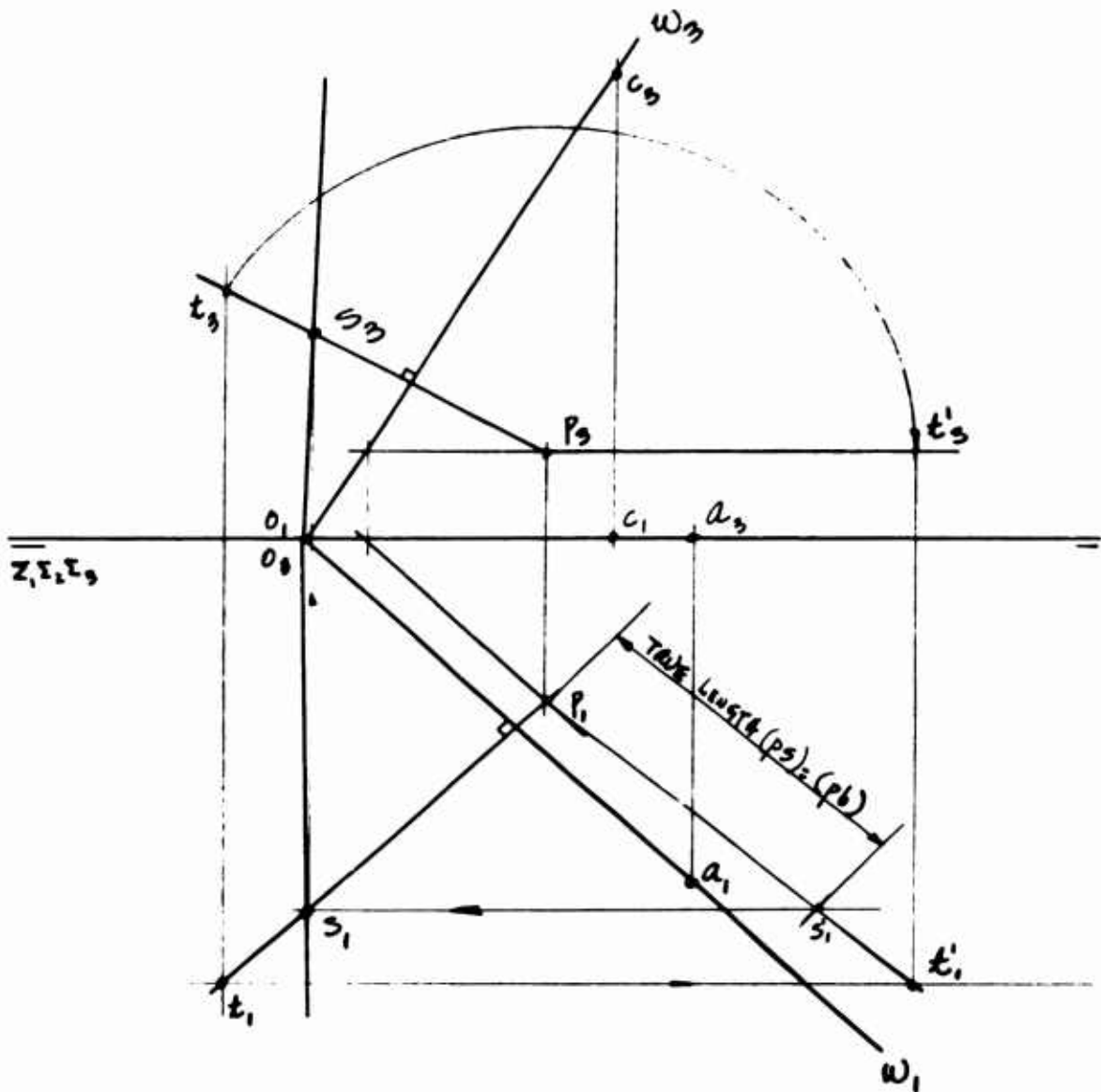


Figure 16

Thus, in figure 16 we can identify a trihedral angle of edges  $\omega_1 \equiv (oa)$ ,  $\omega_3 \equiv (oc)$ , and  $(os)$  whose faces are equal to the trihedral angle  $(\omega_1 \omega_2 \omega_3)$  of the 4-D space (figure 13). This trihedral angle (o-acs) characterizes the position of the trihedral  $(\omega_1 \omega_2 \omega_3)$  until its superimposition with the 3-D space  $\Sigma_2$ .

If we want we may check the true value of the faces of each trihedral and see that they are, in fact, equal. Naturally, we can see that the face  $\omega_1 \omega_3$  is common to both trihedral angles.

The determination of the true value of the faces of the trihedral  $(\omega_1 \omega_2 \omega_3)$  is shown in figure 17.

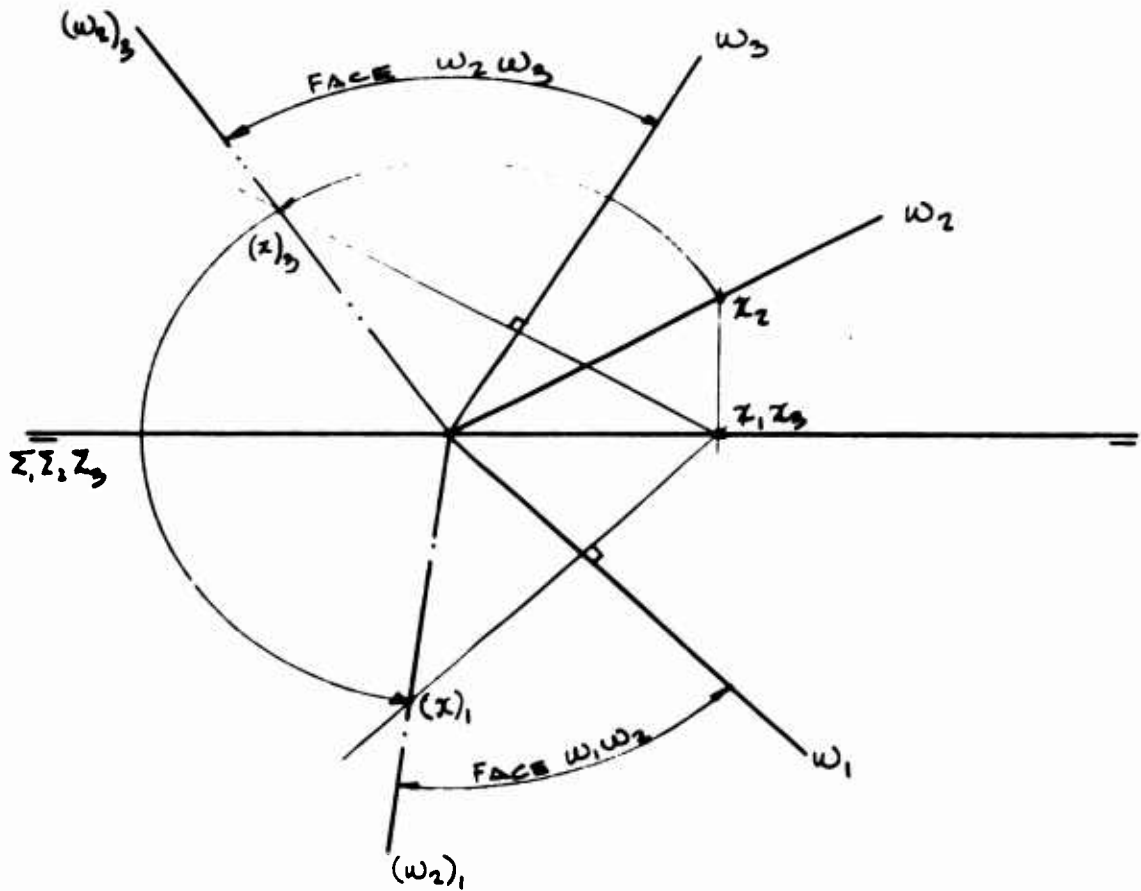


Figure 17

In figure 18 we show the determination of the true value of the faces (o-sc) - corresponding to face  $(w_2 w_3)$  - and (o-sa) - corresponding to face  $(w_1 w_2)$ .

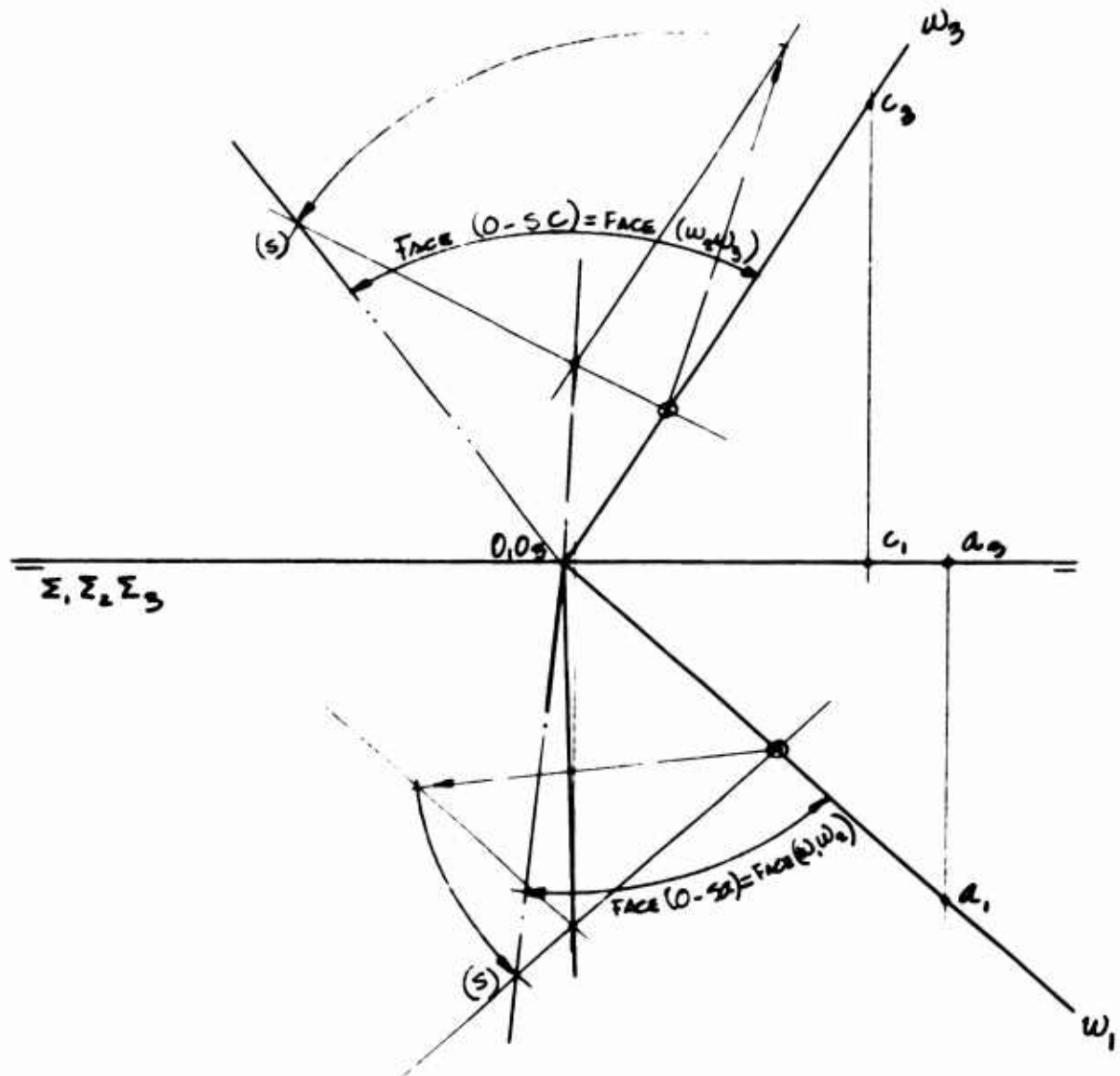


Figure 18 3)

- 3) Consult a treatise on theoretical three-dimensional descriptive geometry for justification of the constructions.

Any point of the 3-D space  $\Omega$  may be referred to the plane  $\omega_1\omega_2$  as it was done with the point (b) and its position, after the rotation, determined in function of the foot of the perpendicular through the point to  $\omega_1\omega_2$  and in function of its distance to that plane. In resume, it becomes a problem of trihedral angle, where are given a face, the projection of a point of the opposite edge on that face, and the distance from the point to the face.

#### FOURTH PROBLEM

"To rotate a point of a 3-D space about one of its traces until it belongs to one of the planes of the 4-D system of reference."

Let  $T$  be a given 3-D space and (o) a point belonging to a plane  $\gamma$  of that 3-D space. Figure 19.

23.

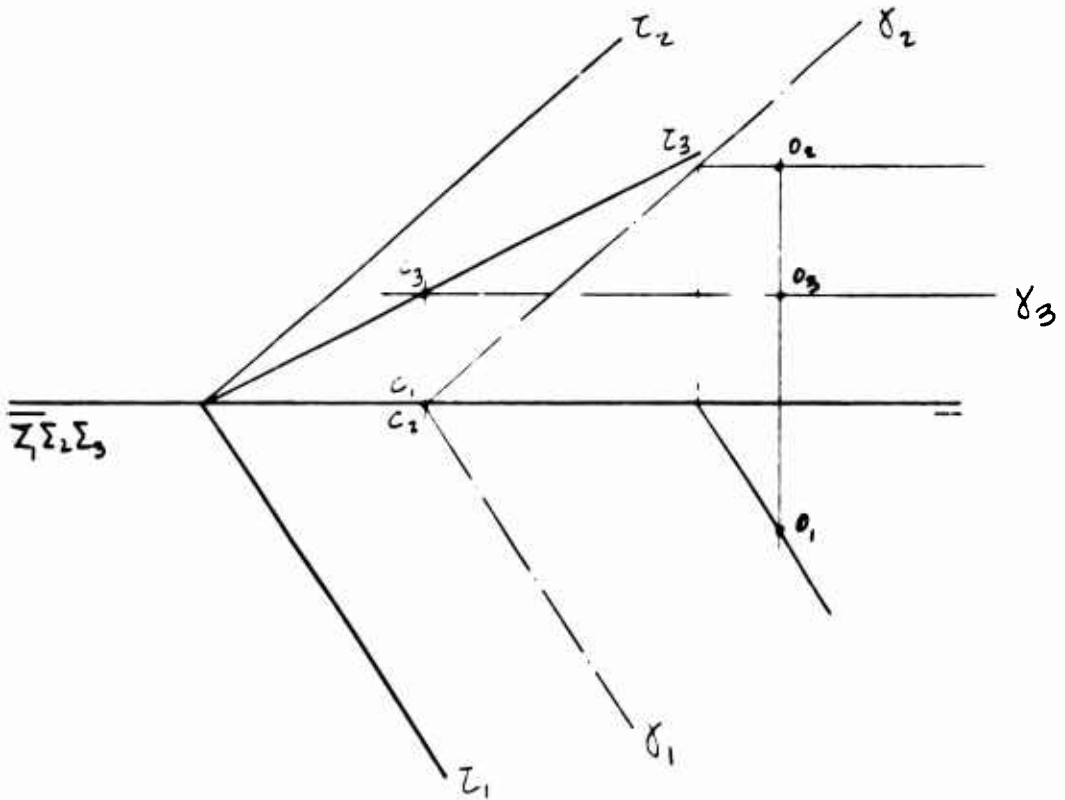


Figure 19

Because of the conditions of belonging between a point and a 3-D space, we may also consider two other planes,  $\alpha$  and  $\beta$ , belonging to the point (o) and to the 3-D space  $\mathcal{T}$ , as indicated in figure 20. Thus, the point (o) is the intersection of planes  $\alpha$ ,  $\beta$ , and  $\delta$ .

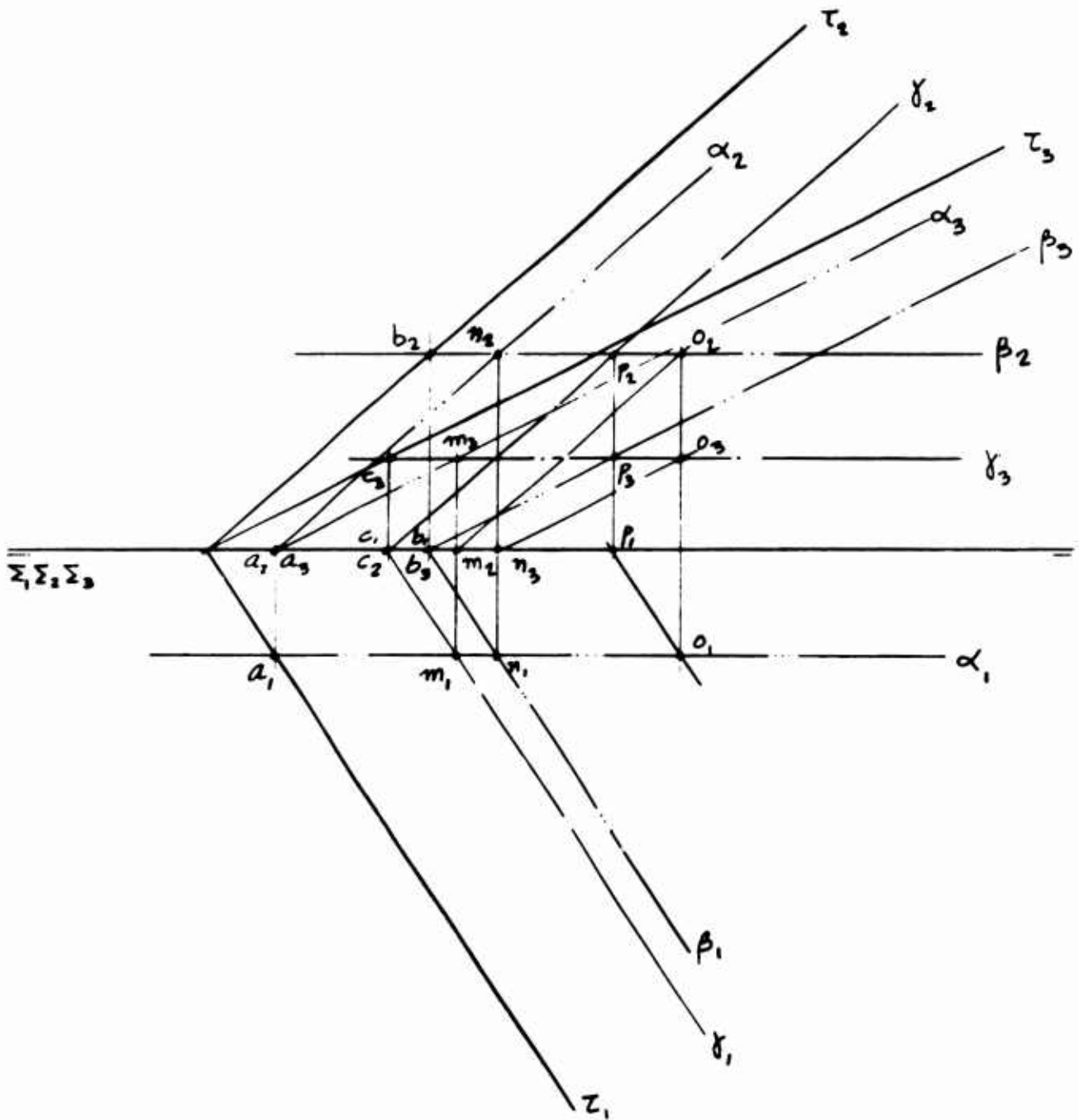


Figure 20

We can write:

Lines (mo) and (no) belong to plane  $\alpha$

$$(mo) // \alpha_2 : (\alpha_2 // \tau_2)$$

$$(no) // \alpha_3 : (\alpha_3 // \tau_3)$$

Lines (po) and (no) belong to plane  $\beta$

$$(po) // \beta_1 : (\beta_1 // \tau_1)$$

$$(no) // \beta_3 : (\beta_3 // \tau_3)$$

Lines (mo) and (po) belong to plane  $\gamma$

$$(mo) // \gamma_2 : (\gamma_2 // \tau_2)$$

$$(po) // \gamma_1 : (\gamma_1 // \tau_1)$$

Therefore,

$$\alpha \times \beta \Rightarrow (no)$$

$$\alpha \times \gamma \Rightarrow (mo)$$

$$\beta \times \gamma \Rightarrow (po)$$

Furthermore, we observe the following conditions of belonging:

Point (n) belongs to  $\gamma_1$  and to  $\alpha_3$ .

Point (n) belongs to  $\beta_1$  and to  $\alpha_2$ .

Point (p) belongs to  $\beta_3$  and to  $\gamma_2$ .

Point (a) belongs to  $\tau_1$  and to plane  $\alpha$ .

Point (b) belongs to  $\tau_2$  and to plane  $\beta$ .

Point (c) belongs to  $\tau_3$  and to plane  $\gamma$ .

If we proceed in obtaining the superimposition of planes  $\tau_1-\tau_2$ ,  $\tau_1-\tau_3$  on plane  $\pi_1$ , by rotating each of them about the line  $\tau_1$ , we arrive to the results shown in figure 21.

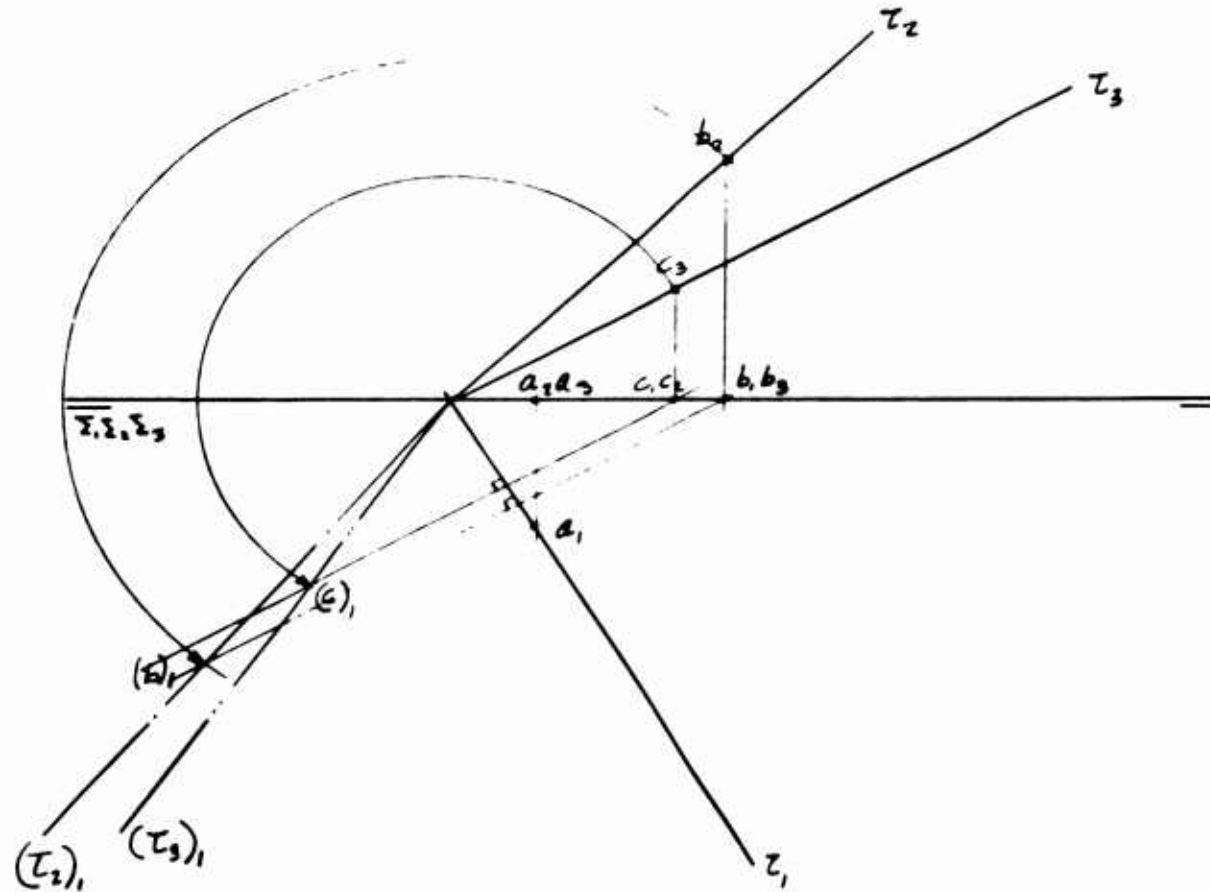


Figure 21

Therefore, due to the conditions of belonging among the points, lines and planes, we obtain the location of these points, lines and planes, when each is superimposed on  $\pi_1$

27.

by rotation about the line  $\tau_1$ . See figure 22.

To obtain the position of  $(o)_1$ , draw perpendicular to  $\tau_1$  through  $o_1$ : from  $(m)_1$ ,  $(n)_1$ , and  $(p)_1$ , draw parallels to  $(\tau_2)_1$ ,  $(\tau_3)_1$ , and  $(\tau_1)_1$ , respectively. All these lines will meet in  $(o)_1$ .

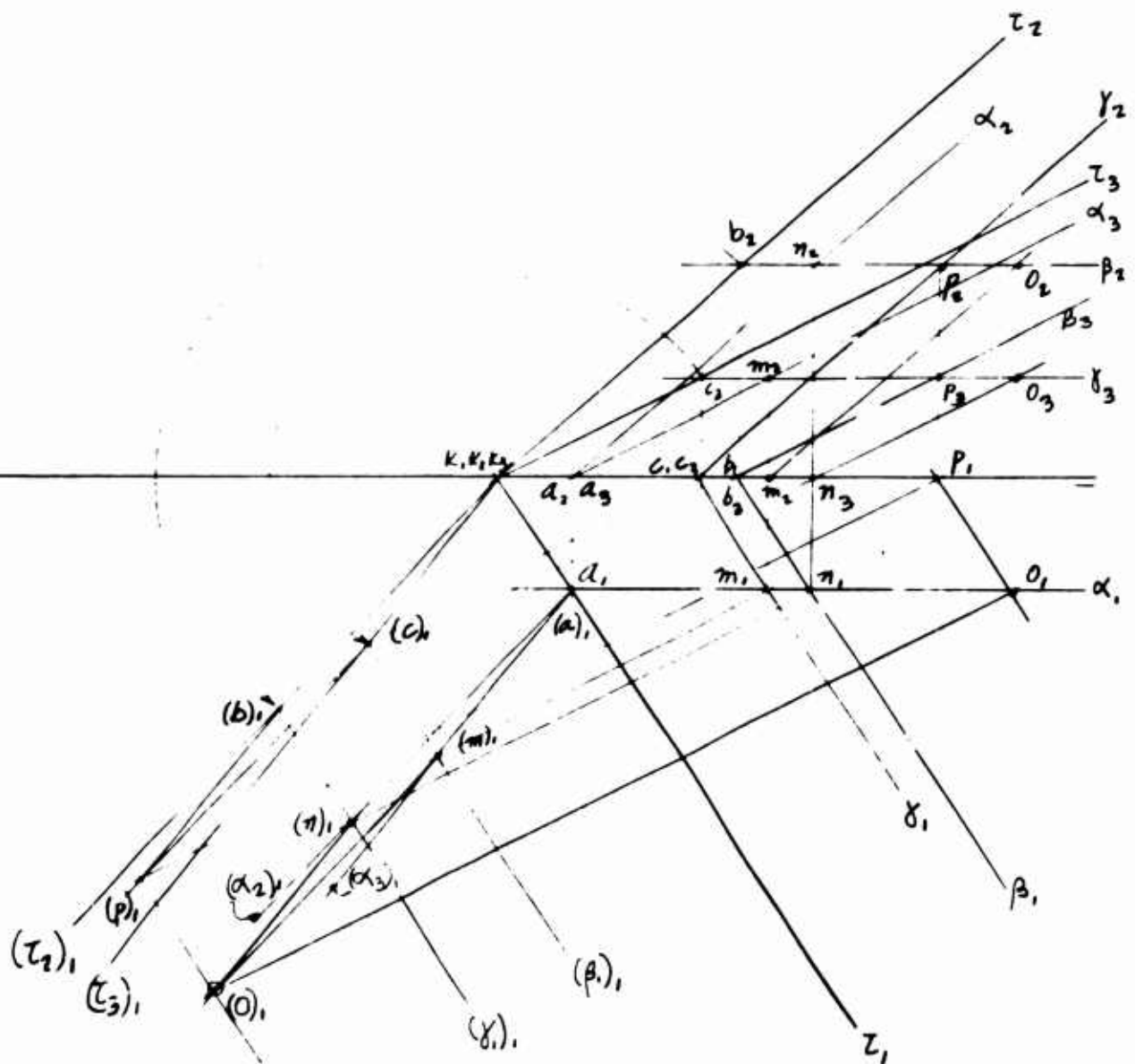


Figure 22

A verification of the construction just outlined is as follows and may be used as a method for determining the position of the points of a plane determined by a trace of the 3-D space and a point. However, if the position of the point is known, in the plane  $\pi_1$ , for example, we will have to make use of the constructions shown in figure 22, in order to determine its projection.

The second method is based on the determination of the true length of the segment  $(ko)$ . This true length will also appear as  $[(k)-(o)_1]$  in the plane  $\pi_1$ . Therefore, to determine the position of  $(o)_1$ , first determine the true length of  $(ko)$  and center in  $(k)$  and radius  $(ko)$ , cut the perpendicular drawn through  $o_1$  to  $\tau_1$ , in the point  $(o)_1$ . Figure 23.

Verification:

$[(a)_1 - (o)_1]$  in figure 22

$[(k) - (o)_1]$  in figure 23

$$[(a)_1 - (o)_1] = [(k) - (o)_1]$$

30.

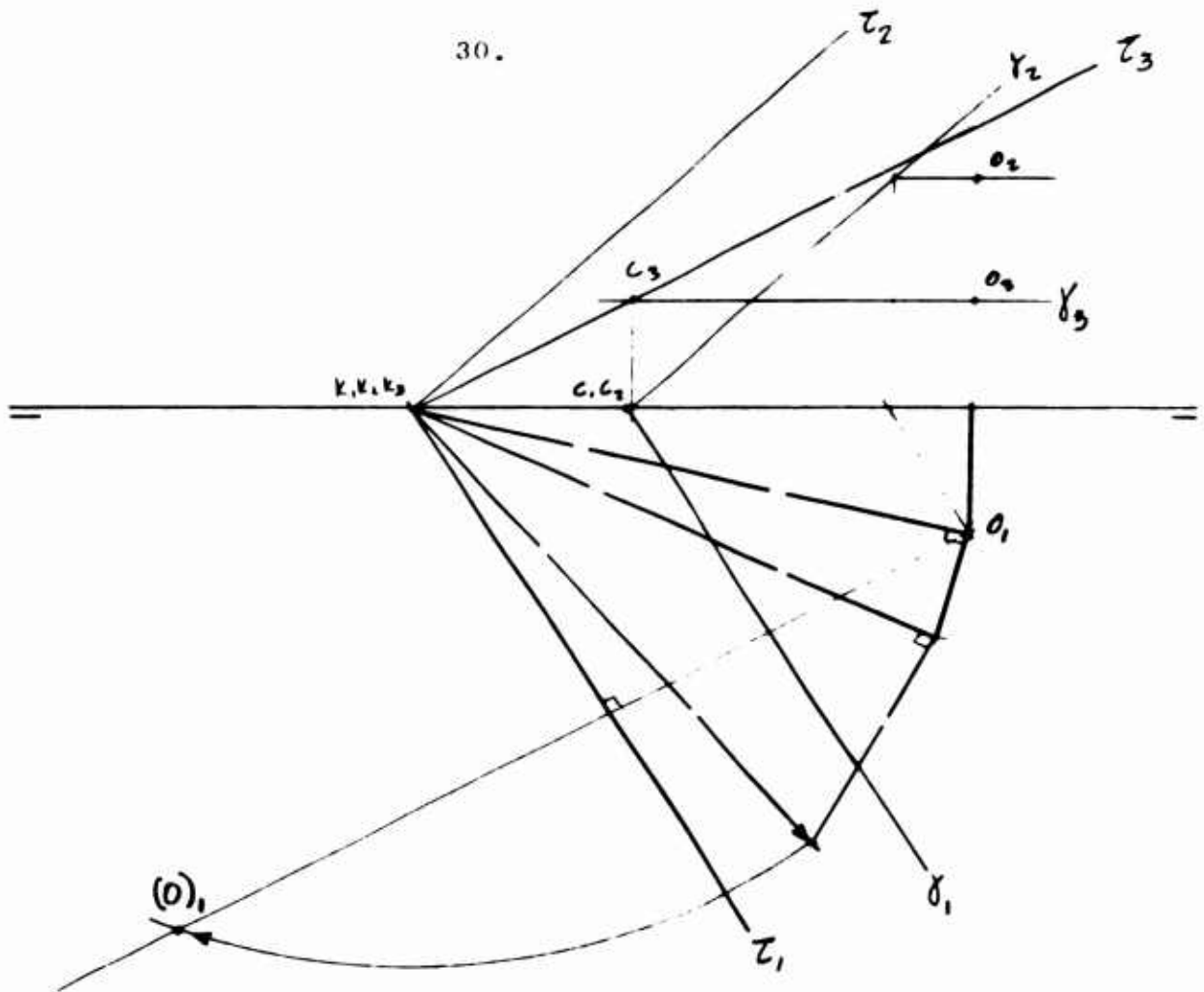


Figure 23

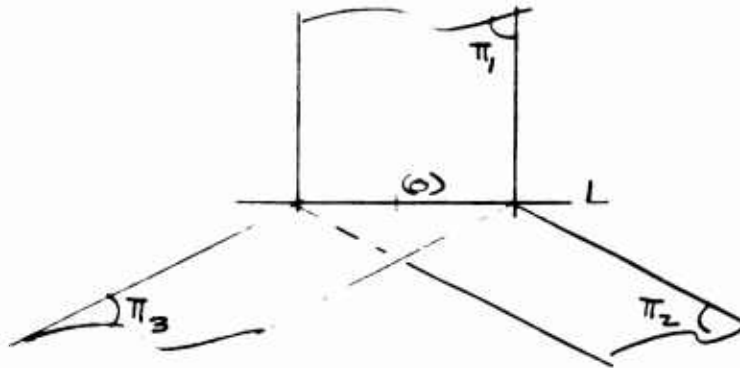
# FIFTH PROBLEM

"In the 4-D system of reference there are six planes, two by two perpendicular, three by three belonging to the same line. Question: what is the section made by a plane belonging to a point of the line of three planes and perpendicular to it?"

SOLUTION

Any problem involving plane sections in the four-dimensional space, ought to be preceded by the section of a 3-D space that belongs to the plane.

Let then the planes  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$ , (figure 24), be the planes of the 4-D system of reference. They belong to the line  $L$ .

Figure 24

Through the point (o) we shall consider a plane  $\alpha$ , perpendicular to  $L$ , and also, a 3-D space  $\Omega$ , belonging to  $\alpha$ . Therefore, line  $L$  is perpendicular to  $\Omega$ .

To determine the section made by  $\alpha$ , we shall, first determine the section made by  $\Omega$  in each of the planes  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$ , that is, in the 3-D spaces  $\Sigma_1$ ,  $\Sigma_2$ , and  $\Sigma_3$

that they determine. According to the study of the representation of a 3-D space, we will obtain three lines of  $\Omega$ , in  $\pi_1, \pi_2, \pi_3$ , all perpendicular to  $L$  and belonging to  $(o)$ . Let these lines be  $\omega_1, \omega_2$ , and  $\omega_3$ .

We can write:

$$\begin{aligned}\Omega \times \Sigma_1 &\Rightarrow \text{plane } \omega_2 - \omega_3 \\ \Omega \times \Sigma_2 &\Rightarrow \text{plane } \omega_1 - \omega_3 \\ \Omega \times \Sigma_3 &\Rightarrow \text{plane } \omega_1 - \omega_2\end{aligned}$$

These are three distinct planes, all belonging to  $(o)$ , two by two perpendicular, and each one perpendicular to a plane  $\pi_i$ ,

$$\begin{aligned}(\omega_1 - \omega_2) &\perp \pi_3 \\ (\omega_1 - \omega_3) &\perp \pi_2 \\ (\omega_2 - \omega_3) &\perp \pi_1\end{aligned}$$

and all perpendicular to  $L$ .

$$\begin{aligned}(\omega_1 - \omega_2) &\perp L \\ (\omega_1 - \omega_3) &\perp L \\ (\omega_2 - \omega_3) &\perp L\end{aligned}$$

Therefore, due to the above results, we conclude that the plane  $\alpha$  is one of these planes. Thus, the section made

by plane  $\alpha$  in the 4-D system, are two of the three lines  $\omega_1, \omega_2, \omega_3$ , depending on the condition of belonging involving  $\alpha$  and one of the three 3-D spaces  $\Sigma_1, \Sigma_2, \Sigma_3$ .

With this result, we want to observe that the observations made in our paper "On the Generation of a Spherical Surface in a Four-Dimensional Space"<sup>3)</sup>, are not correctly stated, since we assumed that the three lines  $\omega_1, \omega_2$ , and  $\omega_3$ , (noted in that paper as OA, OD, OE), belong to one unique plane.

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3) Technical Seminar Series, Report No. 14, December 10, 1964, Department of Graphics and Engineering Drawing, Princeton University.

# Case File

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September 22, 1965.

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Gentlemen:

Enclosed are copies of Special Reports #3, 4, 5, 6, 7, 8  
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Four-Dimensional Descriptive Geometry Rotation -  
Descriptive Solution, March 1965.

Four-Dimensional Descriptive Geometry Problems on  
3-D Spaces, March 1965.

Graphical Plotting of Data. An Application of Three  
and Four Dimensional Theoretical Descriptive Geometries,  
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Four-Dimensional Descriptive Geometry Metric Problems:  
Angles - Descriptive Solution, May 1965.

Four-Dimensional Descriptive Geometry Proposition of a  
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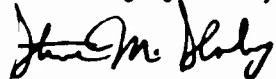
Four-Dimensional Descriptive Geometry Metric Problems:  
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